

THE ESSENTIAL SPECTRUM OF THE LAPLACIAN ON RAPIDLY BRANCHING TESSELLATIONS

MATTHIAS KELLER

ABSTRACT. In this paper we characterize emptiness of the essential spectrum of the Laplacian under a hyperbolicity assumption for general graphs. Moreover we present a characterization for emptiness of the essential spectrum for planar tessellations in terms of curvature.

0. INTRODUCTION AND MAIN RESULTS

The paper is dedicated to investigate the essential spectrum of the Laplacian on graphs. More precisely the purpose is threesome. Firstly we give a comparison theorem for the essential spectra of the Laplacian Δ used in the Mathematical Physics community (see for instance [ASW, AF, AV, Br, CFKS, FHS, GG, Go, Kl, KLPS]) and the combinatorial Laplacian $\tilde{\Delta}$ used in Spectral Geometry (see for instance [DKa, DKe, Fu, Wo1]) on general graphs.

Secondly we consider graphs which are rapidly branching, i.e. the vertex degree is growing uniformly as one tends to infinity. We establish a criterion under which absence of essential spectrum of the Laplacian Δ is completely characterized. This criterion will be positivity of the Cheeger constant at infinity introduced in [Fu], based on [Che, D1]. It turns out that in the case of planar tessellating graphs this positivity will be implied automatically by uniform growth of vertex degree. Moreover we can interpret the rapidly branching property as a uniform decrease of curvature. An immediate consequence is that these operators have no continuous spectrum.

The third purpose is to demonstrate that Δ and $\tilde{\Delta}$ may show a very different spectral behavior. Therefore we discuss a particular class of rapidly branching graphs. This discussion will also prove independence of our assumptions in the results mentioned above. In the following introduction we will give an overview. We refer to Section 1 for precise definitions.

There is a result of H. Donnelly and P. Li [DL] on negatively curved manifolds. It shows that the Laplacian Δ on a rapidly curving manifold has a compact resolvent, i.e. empty essential spectrum.

Theorem (Donnelly, Li) *Let M be a complete, simply connected, negatively curved Riemannian manifold and $K(r) = \sup\{K(x, \pi) \mid d(p, x) \geq r\}$ the sectional curvature for $r \geq 0$, where d is the distance function on the manifold, $p \in M$ and π is a two plane in $T_x M$. If $\lim_{r \rightarrow \infty} K(r) = -\infty$, then Δ on M has no essential spectrum i.e. $\sigma_{\text{ess}}(\Delta) = \emptyset$.*

A remarkable result of K. Fujiwara [Fu] provides an analogue in the graph case for the combinatorial Laplacian $\tilde{\Delta}$.

Theorem (Fujiwara) *Let $G = (V, E)$ be an infinite graph. Then $\sigma_{\text{ess}}(\tilde{\Delta}) = \{1\}$ if and only if $\alpha_\infty = 1$.*

Here α_∞ is a Cheeger constant at infinity. Since the combinatorial Laplacian $\tilde{\Delta}$ is a bounded operator the essential spectrum can not be empty. Yet it shrinks to one point for $\alpha_\infty = 1$.

We will show that an analogue result holds for the Laplacian Δ , which is used in the community of mathematical physicists. Let $G = (V, E)$ be an infinite graph. For compact $K \subset V$ denote by K^c its complement $V \setminus K$ and let

$$m_K = \inf\{\deg(v) \mid v \in K^c\} \quad \text{and} \quad M_K = \sup\{\deg(v) \mid v \in K^c\},$$

where $\deg : V \rightarrow \mathbb{N}$ is the vertex degree. Denote

$$m_\infty = \lim_{K \rightarrow \infty} m_K \quad \text{and} \quad M_\infty = \lim_{K \rightarrow \infty} M_K.$$

In the next section we will be precise about what we mean by the limits. We call a graph *rapidly branching* if $m_\infty = \infty$. We will prove the following theorems.

Theorem 1. *Let G be infinite. For all $\lambda \in \sigma_{\text{ess}}(\Delta)$ it holds*

$$m_\infty \inf \sigma_{\text{ess}}(\tilde{\Delta}) \leq \lambda \leq M_\infty \sup \sigma_{\text{ess}}(\tilde{\Delta})$$

and

$$\inf \sigma_{\text{ess}}(\Delta) \leq \min\{m_\infty, M_\infty \inf \sigma_{\text{ess}}(\tilde{\Delta})\}.$$

In the first statement we have the convention that if $\inf \sigma_{\text{ess}}(\tilde{\Delta}) = 0$ and $m_\infty = \infty$ we set $m_\infty \inf \sigma_{\text{ess}}(\tilde{\Delta}) = 0$. The first part of the theorem shows that the essential spectra of the operators $\tilde{\Delta}$ and Δ correspond in terms of the minimal and maximal vertex degree at infinity. The second part gives two options to estimate the infimum of the essential spectrum of Δ from above.

Our second theorem is the characterization of emptiness of the essential spectrum.

Theorem 2. *Let $G = (V, E)$ be infinite and $\alpha_\infty > 0$. Then $\sigma_{\text{ess}}(\Delta) = \emptyset$ if and only if $m_\infty = \infty$.*

Note that $m_\infty = \infty$ does not imply $\alpha_\infty > 0$ or $\sigma_{\text{ess}}(\Delta) = \emptyset$. This will be discussed in Section 4.

We may interpret $\alpha_\infty > 0$ as an assumption on the graph to be hyperbolic at infinity. (See discussion in [Hi] and the references [GH, Gr, LS] found there.) Moreover the growth of the vertex degree can be interpreted as decrease of the curvature. In this way we may understand Theorem 2 as an analogue of Donnelly and Li for Δ on graphs. For tessellating graphs this analogy will be even more obvious. Since the continuous spectrum of an operator is always contained in the essential spectrum there is an immediate corollary.

Corollary 1. *Let $G = (V, E)$ be infinite, $\alpha_\infty > 0$ and $m_\infty = \infty$. Then Δ has pure point spectrum.*

The class of examples for which [Fu] shows absence of essential spectrum are rapidly branching trees. We will show that the result is also valid for rapidly branching tessellations. We will formulate the statement in terms of the curvature because this makes the analogy to Donnelly and Li more obvious. For this sake we define the combinatorial curvature function $\kappa : V \rightarrow \mathbb{R}$ for a vertex $v \in V$ as it is found in [BP1, BP2, Hi, Wo2] by

$$\kappa(v) = 1 - \frac{\deg(v)}{2} + \sum_{f \in F, v \in f} \frac{1}{\deg(f)},$$

where $\deg(f)$ denotes the number of vertices contained in a face $f \in F$. For compact $K \subset V$ let

$$\kappa_K = \sup\{\kappa(v) \mid v \in K^c\}$$

and $\kappa_\infty = \lim_{K \rightarrow \infty} \kappa_K$. Obviously $\kappa_\infty = -\infty$ is equivalent to $m_\infty = \infty$. Here is our main theorem.

Theorem 3. *Let G be a tessellation. Then $\sigma_{\text{ess}}(\Delta) = \emptyset$ if and only if $\kappa_\infty = -\infty$. Moreover $\kappa_\infty = -\infty$ implies $\sigma_{\text{ess}}(\tilde{\Delta}) = \{1\}$.*

The theorem is a special case of Theorem 2. The hyperbolicity assumption $\alpha_\infty > 0$ follows from $\kappa_\infty = -\infty$ in the case of tessellating graph. In particular it even holds $\alpha_\infty = 1$ whenever the curvature tends uniformly to $-\infty$.

Klassert, Lenz, Peyerimhoff, Stollmann [KLPS] proved the absence of compactly supported eigenfunctions for non-positively curved tessellations. Since $\kappa_\infty = -\infty$ implies non-positive curvature outside of a

certain set K the result applies here. Hence we have pure point spectrum such that all eigenfunctions are either supported in K or on an infinite set.

To end this section we introduce a technical proposition which is used almost throughout all the proofs of the paper. It uses quite standard technics and may be of independent interest. For a linear operator B on a space of functions on V , we write B_K for its restriction to the space of functions on K^c with Dirichlet boundary conditions, where $K \subset V$ is compact set. Let $l^2(V, g)$ be the space of square summable functions with respect to the weight function g and $c_c(V)$ the space of compactly supported functions on V .

Proposition 1. *Let $G = (V, E)$ be infinite and B a self adjoint operator with $c_c(V) \subseteq D(B) \subseteq l^2(V, g)$ which is bounded from below. Then*

$$\begin{aligned} \inf \sigma_{\text{ess}}(B) &= \lim_{K \rightarrow \infty} \inf_{\substack{\varphi \in c_c(V) \\ \text{supp } \varphi \subseteq K^c}} \frac{\langle B\varphi, \varphi \rangle_g}{\langle \varphi, \varphi \rangle_g} = \lim_{K \rightarrow \infty} \inf \sigma(B_K), \\ \sup \sigma_{\text{ess}}(B) &\leq \lim_{K \rightarrow \infty} \sup_{\substack{\varphi \in c_c(V) \\ \text{supp } \varphi \subseteq K^c}} \frac{\langle B\varphi, \varphi \rangle_g}{\langle \varphi, \varphi \rangle_g} = \lim_{K \rightarrow \infty} \sup \sigma(B_K). \end{aligned}$$

If B is bounded, we have equality in the second formula.

The paper is structured as follows. In Section 1 we will define the versions of the Laplacian which appear in different contexts of the literature. We discuss Fujiwara's Theorem which can be understood as a result on compact operators. In Section 2 we prove Proposition 1 and Theorem 1 and 2. In Section 3 we give an estimate of the Cheeger constant at infinity for planar tessellations and prove Theorem 3. Finally in Section 4 we discuss a class of examples which shows that for general graphs Δ and $\tilde{\Delta}$ can have a quite different spectral behavior.

1. THE COMBINATORIAL LAPLACIAN $\tilde{\Delta}$ IN TERMS OF COMPACT OPERATORS

Let $G = (V, E)$ be a connected graph with finite vertex degree in each vertex. For a positive weight function $g : V \rightarrow \mathbb{R}_+$ let

$$\begin{aligned} l^2(V, g) &= \{ \varphi : V \rightarrow \mathbb{R} \mid \langle \varphi, \varphi \rangle_g = \sum_{v \in V} g(v) |\varphi(v)|^2 < \infty \}, \\ c_c(V) &= \{ \varphi : V \rightarrow \mathbb{R} \mid |\text{supp } \varphi| < \infty \} \end{aligned}$$

where supp is the support of a function. For $g = 1$ we write $l^2(V)$. Notice that $l^2(V, g)$ is the completion of $c_c(V)$ under $\langle \cdot, \cdot \rangle_g$. For $g = \deg$ it is clear that $l^2(V, \deg) \subseteq l^2(V)$ and if $\sup_{v \in V} \deg(v) < \infty$ then $l^2(V) = l^2(V, \deg)$. We occasionally write $l^2(G, g)$ for $l^2(V, g)$.

For $\varphi \in c_c(V)$ and $v \in V$ define the operators

$$(A\varphi)(v) = \sum_{u \sim v} \varphi(u) \quad \text{and} \quad (D\varphi)(v) = \deg(v)\varphi(v).$$

The operator A is often called the adjacency matrix. Since we assumed that the graph has no isolated vertices the operator D has a bounded inverse.

The Laplace operator plays an important role in different areas of mathematics. Yet there occur different versions of it. To avoid confusion we want to discuss them briefly. We start with the Laplacian used in the Mathematical Physicist community in the context of Schrödinger operators. For reference see e.g. [CFKS, D1] (and the references there) or in more recent publications like [AF, ASW, AV, Br, D2, FHS, GG, Go, Kl, KLPS].

(1.) The operator $D - A$ defined on $c_c(V)$ yields the following form

$$\langle d\varphi, d\varphi \rangle = \frac{1}{2} \sum_{v \in V} \sum_{u \sim v} |\varphi(u) - \varphi(v)|^2.$$

The self adjoint operator on $l^2(V)$ corresponding to this form will be denoted by Δ . It gives for $\varphi \in D(\Delta)$ and $v \in V$

$$(\Delta\varphi)(v) = \deg(v)\varphi(v) - \sum_{u \sim v} \varphi(u).$$

Notice that Δ is unbounded if there is no bound on the vertex degree.

We next introduce the combinatorial Laplacian. Two unitary equivalent versions are found in the literature. They are given as follows.

(2.) Let

$$\tilde{\Delta} = I - \tilde{A} = I - D^{-1}A$$

be defined on $l^2(V, \deg)$, where I is the identity operator. It is easy to see that $\tilde{\Delta}$ is bounded and self adjoint. For $\varphi \in l^2(V, \deg)$ and $v \in V$ it gives

$$(\tilde{\Delta}\varphi)(v) = \varphi(v) - \frac{1}{\deg(v)} \sum_{u \sim v} \varphi(u).$$

The matrix \tilde{A} is often called the transition matrix. This version of the combinatorial Laplacian can be found for instance in [DKa, DKe, Fu, Wo1] and many others.

(3.) There is a unitary equivalent version as discussed e.g. in [Chu]. Let

$$\hat{\Delta} = I - \hat{A} = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$$

be defined on $l^2(V)$. It gives for $\varphi \in l^2(V)$ and $v \in V$

$$(\hat{\Delta}\varphi)(v) = \varphi(v) - \sum_{u \sim v} \frac{1}{\sqrt{\deg(u)\deg(v)}} \varphi(u).$$

Notice that the operator

$$D_{1,\deg}^{\frac{1}{2}} : l^2(V, \deg) \rightarrow l^2(V), \quad \varphi \mapsto \sqrt{\deg} \cdot \varphi,$$

is an isometric isomorphism and we denote its inverse by $D_{\deg,1}^{-\frac{1}{2}}$. Then

$$\widehat{\Delta} = D_{1,\deg}^{\frac{1}{2}} \widetilde{\Delta} D_{\deg,1}^{-\frac{1}{2}}.$$

Moreover on $c_c(V)$

$$\Delta = D^{\frac{1}{2}} \widehat{\Delta} D^{\frac{1}{2}}.$$

Furthermore we define the Dirichlet restrictions of these operators. For a set $K \subseteq V$ let $P_K : l^2(V, g) \rightarrow l^2(K^c, g)$ be the canonical projection and $i_K : l^2(K^c, g) \rightarrow l^2(V, g)$ its dual operator, which is the continuation by 0 on K . For an operator B on $l^2(V, g)$ we write

$$B_K = P_K B i_K.$$

Hence we can speak of Δ_K , $\widetilde{\Delta}_K$ or $\widehat{\Delta}_K$ on K^c with Dirichlet boundary conditions. Mostly K will be a compact set.

For a graph G and compact $K \subseteq V$ define the *Cheeger constant*, see [Che, DKe, Fu],

$$\alpha_K = \inf_{W \subseteq K^c, |W| < \infty} \frac{|\partial_E W|}{A(W)},$$

where $\partial_E W$ is the set of edges which have one vertex in W and one outside and $A(W) = \sum_{v \in W} \deg(v)$. Let $W \subseteq K^c$, for K compact and χ the characteristic function of W . Two simple calculations, mentioned in [DKa] yield

$$\langle \widetilde{\Delta}_K \chi, \chi \rangle_{\deg} = \langle \Delta_K \chi, \chi \rangle = |\partial_E W|$$

and

$$\langle \chi, \chi \rangle_{\deg} = \langle D \chi, \chi \rangle = A(W).$$

This gives

$$(1) \quad \alpha_K = \inf_{W \subseteq K^c, |W| < \infty} \frac{\langle \widetilde{\Delta}_K \chi_W, \chi_W \rangle_{\deg}}{\langle \chi_W, \chi_W \rangle_{\deg}}.$$

The set $K(V)$ of compact subsets of V is a net under the partial order \subseteq . We say a function $F : K(V) \rightarrow \mathbb{R}$, $K \mapsto F_K$ converges to $F_\infty \in \mathbb{R}$ if for all $\epsilon > 0$ there is a $K_\epsilon \in K(V)$ such that $|F_K - F_\infty| < \epsilon$ for all $K \supseteq K_\epsilon$. We then write $\lim_{K \rightarrow \infty} F_K = F_\infty$. With this convention we define the *Cheeger constant at infinity* like [Fu] by

$$\alpha_\infty = \lim_{K \rightarrow \infty} \alpha_K.$$

The limit always exists since $\alpha_K \leq \alpha_L \leq 1$ for compact $K \subseteq L \subseteq V$. Therefore we can think of taking the limit over distance balls of an arbitrary vertex.

The next part is dedicated to a discussion of [Fu]. We will look at the result from the perspective of compact operators. The proof is based on two propositions which hold for general graphs. We present them here as norm estimates on the transition matrix. The essential part for the proof of the first proposition was remarked in [DKa].

Proposition 2. *For any compact set $K \subseteq V$*

$$\|\tilde{A}_K\| \geq 1 - \alpha_K.$$

In particular $\inf \sigma(\tilde{\Delta}_K) \leq \alpha_K$.

Proof By equation (1) we receive $\inf \sigma(\tilde{\Delta}_K) \leq \alpha_K$. Since \tilde{A}_K is self adjoint we get $\inf \sigma(\tilde{\Delta}_K) = \inf \sigma(I - \tilde{A}_K) = 1 - \sup \sigma(\tilde{A}_K) = 1 - \|\tilde{A}_K\|$ and thus $\|\tilde{A}_K\| \geq 1 - \alpha_K$. \square

Proposition 3. *For any compact set $K \subseteq V$*

$$\|\tilde{A}_K\| \leq \sqrt{1 - \alpha_K^2}.$$

The second proposition is derived from the proof of the Theorem in [DKe]. Alternatively it may be derived from Proposition 1 in [Fu]. The essential of the proof of this proposition goes back to Dodziuk, Kendall but the statement can be found explicitly in Fujiwara. We will use it later, so we state it here as a Theorem.

Theorem 4. *For $K \subseteq V$ compact*

$$1 - \sqrt{1 - \alpha_K^2} \leq \tilde{\Delta}_K \leq 1 + \sqrt{1 - \alpha_K^2}.$$

Remark. (1.) For $K = \emptyset$ we get these estimates for the operator $\tilde{\Delta}$. We can also take the limits over K . This is one implication in Fujiwara's Theorem, since $\sigma_{\text{ess}}(\tilde{\Delta}) = \sigma_{\text{ess}}(\tilde{\Delta}_K) \subseteq \sigma(\tilde{\Delta}_K)$.

(2.) Since the operators $\tilde{\Delta}$ and $\hat{\Delta}$ are unitary equivalent, similar statements hold for $\hat{\Delta}$ and \hat{A} .

The essential parts of the next theorem are already found in [Fu]. The implications (i.) \Rightarrow (ii.), (iii.) \Rightarrow (ii.) and (iii.) \Leftrightarrow (iv.) are minor extensions.

Theorem 5. *Let G be infinite. The following are equivalent.*

- (i.) $\sigma_{\text{ess}}(\tilde{\Delta})$ consists of one point.
- (ii.) $\sigma_{\text{ess}}(\tilde{\Delta}) = \{1\}$.
- (iii.) \tilde{A} is compact.
- (iv.) $\lim_{K \rightarrow \infty} \|\tilde{A}_K\| = 0$.
- (v.) $\alpha_\infty = 1$.

Proof The implication (i.) \Rightarrow (ii.) is a consequence of Proposition 1 and Theorem 4. The implication (ii.) \Rightarrow (i.) is trivial. Furthermore (ii.) is equivalent to $\sigma_{\text{ess}}(\tilde{A}) = \{0\}$ which is equivalent to (iii.). Assume (iii.). Let (K_n) be a growing sequence of compact sets. Choose $f_n \in l^2(K_n^c, \text{deg})$, $\|f_n\|_{\text{deg}} = 1$ such that $2\|\tilde{A}_{K_n}f_n\|_{\text{deg}} \geq \|\tilde{A}_{K_n}\|$. Because f_n is supported on K_n^c the sequence (f_n) tends weakly to 0 as $n \rightarrow \infty$. The compactness of \tilde{A} implies $\lim \| \tilde{A}i_n f_n \|_{\text{deg}} = 0$ and thus $\lim \|\tilde{A}_{K_n}\| = 0$, which is (iv.). We assume (iv.), take the limit over K in Proposition 2 and conclude (v.). Suppose (v.). For compact K we have $\tilde{A} = i_{K^c} \tilde{A}_{K^c} P_{K^c} + i_K \tilde{A}_K P_K + C_K$, where C_K is a compact operator. By Proposition 3 we have $\lim \|\tilde{A}_K\| \leq 0$. Moreover \tilde{A}_{K^c} is compact, since K is compact. Thus \tilde{A} is the norm limit of compact operators and hence compact, which is (iii.). \square

Remark. We may think of the problem in an alternative way. For compact $K \subseteq V$ let $G_K = (V_K, E_K)$ be the graph induced by the vertex set K^c , added by loops in the following way. To each vertex $v \in K^c$ we add as many loops as there are edges in $\partial_E K = \partial_E K^c$ which contain v . (We say an edge is a loop if its beginning and end vertex coincides.) We can define projections and embeddings for $l^2(G, g)$ and $l^2(G_K, g)$ as above. Note that the projected operators from $l^2(V, g)$ to $l^2(K^c, g)$ and $l^2(G, g)$ to $l^2(G_K, g)$ are unitary equivalent. Thus we can separate the proof explicitly in graph and operator theory. Proposition 2 and 3 hold for general graphs in particular also for G_K . On the other hand Theorem 5 is only operator theory, which uses the estimates on the operator norm of \tilde{A}_K .

2. THE ESSENTIAL SPECTRUM OF Δ

In this section we compare the operators $\tilde{\Delta}$ and Δ . We will establish bounds on the essential spectrum of Δ by bounds obtained for $\tilde{\Delta}$. Therefore we will firstly prove Proposition 1. Then we prove two propositions which estimate the infimum of the essential spectrum of Δ from below and above. This will be the ingredients for the proofs of Theorem 1 and 2.

Proof of Proposition 1. Without loss of generality we can assume $B \geq 0$. Let $\lambda_0 = \inf \sigma_{\text{ess}}(B)$. Because $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(B_K) \subseteq \sigma(B_K)$ it holds $\lambda_0 \in \sigma(B_K)$ for any compact $K \subset V$. To show the other direction we prove that if there is an $\lambda \in \sigma(B_K) \setminus \sigma_{\text{ess}}(B)$ then there is $L_0 \supset K$ such that $\lambda \notin \sigma(B_{L_0})$. It follows $\lambda \notin \sigma(B_L)$ for $L \supseteq L_0$, since $\inf \sigma(B_L) \geq \inf \sigma(B_{L_0})$ for $L \supseteq L_0$. For compact $K \subset V$ let $\lambda \in \sigma(\Delta_K)$ such that $\lambda < \lambda_0$. Choose λ_1 such that

$$\lambda < \lambda_1 < \lambda_0.$$

The spectral projection $E_{]-\infty, \lambda_1]}$ is a finite rank operator since $B \geq 0$. Let f_1, \dots, f_n be an orthonormal basis of the finite dimensional subspace $E_{]-\infty, \lambda_1]} l^2(V, g)$. Now for arbitrary $\epsilon > 0$ choose a compact $L_\epsilon \subset V$ so large that for $L \supseteq L_\epsilon$

$$\max_{j=1, \dots, n} \|P_L f_j\|_g^2 \leq \epsilon.$$

Let $L \supseteq L_\epsilon$. For $\varphi \in l^2(L^c, g)$ with $\|\varphi\|_g = 1$ there are $\beta_1, \dots, \beta_n \in \mathbb{R}$ with $\beta_1^2 + \dots + \beta_n^2 \leq 1$ such that $E_{]-\infty, \lambda_1], L} \varphi = \beta_1 P_L f_1 + \dots + \beta_n P_L f_n$, where $E_{]-\infty, \lambda_1], L} = P_L E_{]-\infty, \lambda_1]} i_L$. Remember P_L was the projection of $l^2(V, g)$ onto $l^2(L^c, g)$ and i_K its dual. Thus

$$(2) \quad \|E_{]-\infty, \lambda_1], L} \varphi\|_g^2 = \beta_1^2 \|P_L f_1\|_g^2 + \dots + \beta_n^2 \|P_L f_n\|_g^2 \leq \epsilon.$$

Now let $\psi \in l^2(L^c, g)$ such that $\langle B_L \psi, \psi \rangle_g \leq (\inf \sigma(B_L) + \epsilon) \langle \psi, \psi \rangle_g$ and let $d\rho_\psi(\cdot) = d\langle B_L E_{]-\infty, \cdot], L} \psi, E_{]-\infty, \cdot], L} \psi \rangle_g$ be a spectral measure of B_L . Then

$$\begin{aligned} \langle B_L \psi, \psi \rangle_g &= \langle B_L E_{]-\infty, \lambda_1], L} \psi, E_{]-\infty, \lambda_1], L} \psi \rangle_g \\ &\quad + \langle B_L E_{] \lambda_1, \infty[, L} \psi, E_{] \lambda_1, \infty[, L} \psi \rangle_g \\ &\geq \int_{] \lambda_1, \infty[} t \, d\rho_\psi(t) \\ &\geq \lambda_1 \int_{] \lambda_1, \infty[} 1 \, d\rho_\psi(t) \\ &= \lambda_1 (\langle \psi, \psi \rangle_g - \langle E_{]-\infty, \lambda_1], L} \psi, E_{]-\infty, \lambda_1], L} \psi \rangle_g) \\ &\geq \lambda_1 (1 - \epsilon) \langle \psi, \psi \rangle_g. \end{aligned}$$

In the second step we used that B is positive and in the fifth step equation (2). Now we choose $\delta > 0$ such that $\lambda + \delta < \lambda_1$. Moreover let

$$\epsilon = \frac{\lambda_1 - (\lambda + \delta)}{\lambda_1 + 1}$$

and $L_0 = L_\epsilon$. By our choice of ψ and ϵ we get for all $L \supseteq L_0$

$$\inf \sigma(B_L) \geq \frac{\langle B_L \psi, \psi \rangle_g}{\langle \psi, \psi \rangle_g} - \epsilon \geq \lambda_1 (1 - \epsilon) - \epsilon = \lambda + \delta > \lambda.$$

If the operator B is bounded, we can do a similar estimate from above. Otherwise it still holds $\sup \sigma_{\text{ess}}(B) = \sup \sigma_{\text{ess}}(B_K) \leq \sup \sigma(B_K)$. \square

Since $\tilde{\Delta}$ and $\hat{\Delta}$ are unitary equivalent it makes no difference to compare to operators $\tilde{\Delta}$ and Δ or the operators $\hat{\Delta}$ and Δ . Yet Δ and $\hat{\Delta}$ are defined on the same space, so it seems to be easier with notation to compare them. However to do this the following identity is vital. For $\varphi \in c_c(K^c)$ one can calculate

$$(3) \quad \frac{\langle \Delta_K \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} = \frac{\langle D_K^{\frac{1}{2}} \hat{\Delta}_K D_K^{\frac{1}{2}} \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} = \frac{\langle \hat{\Delta}_K D_K^{\frac{1}{2}} \varphi, D_K^{\frac{1}{2}} \varphi \rangle}{\langle D_K^{\frac{1}{2}} \varphi, D_K^{\frac{1}{2}} \varphi \rangle} \frac{\langle D_K \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle}.$$

Proposition 4. *Let G be infinite. Then for $\lambda \in \sigma_{\text{ess}}(\Delta)$*

$$m_\infty \inf \sigma_{\text{ess}}(\widehat{\Delta}) \leq \lambda \leq M_\infty \sup \sigma_{\text{ess}}(\widehat{\Delta}).$$

Proof Let $K \subset V$ be compact. By equation (3) we have for $\varphi \in c_c(K^c)$

$$\frac{\langle \Delta_K \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \geq \frac{\langle \widehat{\Delta}_K D_K^{\frac{1}{2}} \varphi, D_K^{\frac{1}{2}} \varphi \rangle}{\langle D_K^{\frac{1}{2}} \varphi, D_K^{\frac{1}{2}} \varphi \rangle} \inf_{v \in \text{supp } \varphi} \deg(v).$$

For every $\psi \in c_c(K^c)$ there is an $\varphi \in c_c(K^c)$ such that $\psi = D_K^{\frac{1}{2}} \varphi$. Furthermore $c_c(K^c)$ is dense in the domain of Δ_K and so we conclude

$$\inf \sigma_{\text{ess}}(\Delta) = \inf \sigma_{\text{ess}}(\Delta_K) \geq m_\infty \inf \sigma(\widehat{\Delta}_K).$$

By Proposition 1 this yields the lower bound. If $M_\infty = \infty$ the upper bound is infinity. Otherwise by equation (3) $\sup \sigma(\Delta_K) \leq M_\infty \sup \sigma(\widehat{\Delta}_K)$ and again by Proposition 1 we the upper bound. \square

Proposition 5. *Let G be infinite. Then*

$$\inf \sigma_{\text{ess}}(\Delta) \leq \min\{m_\infty, M_\infty \inf \sigma_{\text{ess}}(\widehat{\Delta})\}.$$

Proof Let $v_n \in V$, $n \in \mathbb{N}$ be pairwise distinct such that $\deg(v_n) \leq m_\infty$. Moreover let χ_n the characteristic function of v_n . For K compact such that $v_n \in K^c$ it holds

$$\inf_{\varphi \in c_c(K^c)} \frac{\langle \Delta_K \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \leq \langle \Delta_K \chi_n, \chi_n \rangle = \deg(v_n) \leq m_\infty.$$

By Proposition 1 we have $\inf \sigma_{\text{ess}}(\Delta) \leq m_\infty$. On the other hand we have by equation (3) $\inf \sigma(\Delta_K) \leq M_K \inf \sigma(\widehat{\Delta}_K)$. By Proposition 1 we get $\inf \sigma_{\text{ess}}(\Delta) \leq M_\infty \inf \sigma_{\text{ess}}(\widehat{\Delta})$. \square

Proof of Theorem 1. Remember the operators $\widetilde{\Delta}$ and $\widehat{\Delta}$ are unitary equivalent. Thus $\sigma_{\text{ess}}(\widetilde{\Delta}) = \sigma_{\text{ess}}(\widehat{\Delta})$. The Theorem follows from Proposition 4 and 5. \square

Proof of Theorem 2. By Theorem 4 we have

$$\inf \sigma(\widehat{\Delta}_K) \geq 1 - \sqrt{1 - \alpha_K^2} \geq 0.$$

Thus by taking the limits Proposition 1 yields $\inf \sigma_{\text{ess}}(\widehat{\Delta}) > 0$ if $\alpha_\infty > 0$. Propositions 4 and 5 give the desired result. \square

Remark. Define $H^1(V) \subseteq l^2(V)$ as the subspace consisting of all f with

$$\|f\|_{H^1} = \|f\| + \langle df, df \rangle^{\frac{1}{2}} < \infty,$$

where the second term in the sum is the form of Δ which was defined in Section 1. Let $j : H^1(V) \rightarrow l^2(V)$ be the canonical inclusion. Then

$\sigma_{\text{ess}}(\Delta) = \emptyset$ if and only if j is compact. This can easily be seen by the fact that $\sigma_{\text{ess}}(\Delta) = \emptyset$ if and only if $(\Delta^{\frac{1}{2}} + I)^{-1}$ is compact.

3. RAPIDLY BRANCHING TESSELLATIONS

In [Fu] the discussed examples are rapidly branching trees. Fujiwara showed that for trees $\alpha_{\infty} = 1$ is implied by $m_{\infty} = \infty$. Therefore by Theorem 2 we have $\sigma_{\text{ess}}(\Delta) = \emptyset$ in the case of trees. In this section we want to extend the class of examples to tessellations. We do this by showing that $\alpha_{\infty} = 1$ is implied by $m_{\infty} = \infty$ for tessellations as well. For planar graphs tessellations are quite well understood. We restrict ourselves to the definitions and refer the reader to [BP1, BP2] and the references contained in there.

Let $G = (V, E)$ be a planar, locally finite graph without loops and multiple edges, embedded in \mathbb{R}^2 . We denote the set of closures of the connected components in $\mathbb{R}^2 \setminus \bigcup_{e \in E} e$ by F and call the elements of F the faces of G . We may write $G = (V, E, F)$. A union of faces is called a *polygon* if it is homeomorphic to a closed disc in \mathbb{R}^2 and its boundary is a closed path of edges without repeated vertices. The graph G is called a *tessellation* or *tessellating* if the following conditions are fulfilled.

- i.) Any edge is contained in precisely two different faces.
- ii.) Two faces are either disjoint or intersect in a unique edge or vertex.
- iii.) All faces are polygons.

Note that a tessellating graph is always infinite. From now on let $G = (V, E, F)$ be tessellating. For a set $W \subseteq V$ let $G_W = (W, E_W, F_W)$ be the induced subgraph, which is the graph with vertex set W and the edges of E which have two vertices in W . Euler's formula states for a connected finite subgraph G_W

$$(4) \quad |W| - |E_W| + |F_W| = 2.$$

Observe that the 2 on the right hand side occurs since we also count the unbounded face. Euler's formula is quite mathematical folklore, nevertheless a proof can be found for instance in [Bo]. We denote by $\partial_F W$ the set of faces in F which contain an edge of $\partial_E W$. In fact each face in $\partial_F W$ contains at least two edges in $\partial_E W$. Therefore $|\partial_F W| \leq |\partial_E W|$ can be checked easily. Moreover we define for finite $W \subseteq V$ the *inner degree* of a face $f \in F$ by

$$\deg_W(f) = |f \cap W|$$

Finally let $C(W)$ be the number of connected components in $G_{V \setminus W}$. Loosely speaking it is the number of holes in G_W .

We need two important formulas which hold for arbitrary finite subgraphs $G_W = (W, E_W, F_W)$ of G . Recall $A(W) = \sum_{v \in W} \deg(v)$. The first formula can be easily rechecked. It reads

$$(5) \quad A(W) = 2|E_W| + |\partial_E W|.$$

As for the second formula note that F_W has faces which are not in F . Nevertheless

$$|F_W| - C(W) = |F_W \cap F|.$$

This is the number of bounded faces which are enclosed by edges of E_W . Thus sorting the following sum over vertices according to faces gives the second formula

$$(6) \quad \sum_{v \in W} \sum_{f \in F, f \ni v} \frac{1}{\deg(f)} = |F_W| - C(W) + \sum_{f \in \partial_F W} \frac{\deg_W(f)}{\deg(f)}.$$

Lemma 1. *Let $G = (V, E, F)$ be a tessellating graph. Then for a finite and connected set $W \subseteq V$*

$$|\partial_E W| \geq A(W) - 6(|W| + C(W) - 2).$$

Proof By the tessellating property we have

$$\sum_{f \in \partial_F W} \deg_W(f) \geq |\partial_E W|.$$

Moreover $\deg(f) \geq 3$ for $f \in F$. Combining this with equation (6) we obtain

$$\begin{aligned} |F_W| &\leq \frac{1}{3} \left(\sum_{v \in W} \sum_{f \ni v} 1 - \sum_{f \in \partial_F W} \deg_W(f) \right) + C(W) \\ &\leq \frac{1}{3} (A(W) - |\partial_E W|) + C(W). \end{aligned}$$

By this estimate, Euler's formula (4) and equation (5) we obtain

$$2 \leq |W| - \frac{1}{6} (A(W) - |\partial_E W|) + C(W),$$

which yields the Lemma. \square

Now we give an estimate from below of the Cheeger constant at infinity.

Proposition 6. *Let G be tessellating. Then*

$$\alpha_\infty \geq 1 - \lim_{K \rightarrow \infty} \sup_{v \in K^c} \frac{6}{\deg(v)}.$$

Proof We assume w.l.o.g. that the compact sets K are distance balls. To calculate α_K we can restrict ourselves to finite sets $W \subset V$, which are connected. Otherwise we find a connected component W_0 of W such that $|\partial_E W_0|/A(W_0) \leq |\partial_E W|/A(W)$. Moreover we can choose W such that $C(W) \leq 2$. Otherwise we find a superset W_1 of W such that

$$|\partial_E W_1|/A(W_1) \leq |\partial_E W|/A(W).$$

Obviously $A(W) \geq |W| \inf_{v \in W} \deg(v)$. By Lemma 1

$$\frac{|\partial_E W|}{A(W)} \geq \frac{A(W) - 6|W|}{A(W)} \geq 1 - \frac{6}{\inf_{v \in W} \deg(v)}.$$

Hence we have $\alpha_K \geq 1 - \sup_{v \in K^c} 6/\deg(v)$. We obtain the result by taking the limit over all compact sets. \square

Remark. The relation between curvature and the Cheeger constant can be presented in more detail than we need it for our purpose here. See therefore [Ke, KP].

Proof of Theorem 3. Let $\kappa_\infty = -\infty$. This is obviously equivalent to $m_\infty = \infty$ which implies $\alpha_\infty = 1$ by Proposition 6. Thus by Theorem 5 and Theorem 1 we obtain $\sigma_{\text{ess}}(\tilde{\Delta}) = \{1\}$ and $\sigma_{\text{ess}}(\Delta) = \emptyset$. On the other hand Proposition 5 tells us that $\sigma_{\text{ess}}(\Delta) = \emptyset$ implies $m_\infty = \infty$ and thus $\kappa_\infty = -\infty$. \square

Remark. The implication that $\sigma_{\text{ess}}(\Delta) = \emptyset$ follows from $\kappa_\infty = -\infty$ can be also obtained on an alternative way. Higuchi [Hi] and Woess [Wo2] showed independently that $\alpha_K > 0$ whenever $\kappa_K < 0$ for $K = \emptyset$. Since $m_\infty = \infty$ is implied by $\kappa_\infty = -\infty$ we can apply Theorem 1 immediately.

4. A FURTHER CLASS OF RAPIDLY BRANCHING GRAPHS

In this section we want to discuss a class of examples which demonstrates that Δ and $\tilde{\Delta}$ can show very different spectral phenomena. In particular this examples prove the independence of our assumptions in Theorem 2.

Let $G(n) = (V(n), E(n))$ be the full graph with n vertices. For $\gamma \geq 0$ and $c \geq 1$ let

$$N_{\gamma,c} : \mathbb{N} \rightarrow \mathbb{N}, \quad n \mapsto n \lceil cn^\gamma \rceil,$$

where $\lceil x \rceil$ is the the smallest integer bigger than $x \in \mathbb{R}$. Denote $N_1 = 1$, $N_2 = \max\{\lceil c \rceil, 2\}$ and for $k \geq 3$

$$N_k = N_{\gamma,c}(N_{k-1}).$$

We construct the graph $G_{\gamma,c}$ as follows. We start with connecting the vertex in $G(N_1)$ with each vertex in $G(N_2)$. We proceed by connecting each vertex in $G(N_k)$ uniquely with $\lceil cN_k^\gamma \rceil$ vertices in $G(N_{k+1})$ for $k \in \mathbb{N}$. Obviously $G_{\gamma,c}$ is rapidly branching whenever $\gamma > 0$ or $c > 1$, in fact $N_k \geq 2^{k-1}$. From another point of view $G_{\gamma,c}$ is a 'tree' of branching number $\lceil cN_k^\gamma \rceil$ in the k -th generation, where we connected the vertices of each generation with one another. The next theorem shows a scheme

of the quite different behavior of the sets $\sigma_{\text{ess}}(\Delta)$ and $\sigma_{\text{ess}}(\tilde{\Delta})$ for the graphs $G_{\gamma,c}$.

Theorem 6. *For $\gamma \geq 0$ and $c \geq 1$ let $G_{\gamma,c}$ be as above.*

*If $\gamma = 0$ then $\alpha_\infty = 0$, $\inf \sigma_{\text{ess}}(\tilde{\Delta}) = 0$ and $\inf \sigma_{\text{ess}}(\Delta) \leq [c]$.
 If $\gamma \in]0, 1[$ then $\alpha_\infty = 0$, $\inf \sigma_{\text{ess}}(\tilde{\Delta}) = 0$ and $\sigma_{\text{ess}}(\Delta) = \emptyset$.
 If $\gamma = 1$ then $\alpha_\infty = \frac{c}{1+c}$, $\inf \sigma_{\text{ess}}(\tilde{\Delta}) \in]0, 1[$ and $\sigma_{\text{ess}}(\Delta) = \emptyset$.
 If $\gamma > 1$ then $\alpha_\infty = 1$, $\sigma_{\text{ess}}(\tilde{\Delta}) = \{1\}$ and $\sigma_{\text{ess}}(\Delta) = \emptyset$.*

As mentioned above all graphs $G_{\gamma,c}$ are rapidly branching if $\gamma > 0$ or $c > 1$. The theorem shows the independence of our assumptions and thus optimality of the result. More precisely the case $\gamma = 0$ shows that $m_\infty = \infty$ alone does not imply $\sigma_{\text{ess}}(\Delta) = \emptyset$. On the other hand the case $\gamma \in]0, 1[$ makes clear that $\sigma_{\text{ess}}(\Delta) = \emptyset$ does not imply $\alpha_\infty > 0$. Moreover when $\gamma = 1$ we see that $m_\infty = \infty$ and $\alpha_\infty > 0$ does not imply $\alpha_\infty = 1$. The last case is an example where $\sigma_{\text{ess}}(\tilde{\Delta}) = \{1\}$ and $\sigma_{\text{ess}}(\Delta) = \emptyset$ like in the case of trees and tessellations.

For a graph G denote by B_n the set of vertices which have distance $n \in \mathbb{N}$ or less from a fixed vertex $v_0 \in V$. In our context choose v_0 as the unique vertex in $G(N_1)$.

The intuition behind the theorem is as follows. Let $S_{n,k} = B_n \setminus B_k$, $n > k$ and $\chi = \chi_{S_{n,k}}$ its characteristic function. Then one can calculate

$$\frac{\langle \Delta_{B_k} \chi, \chi \rangle}{\langle \chi, \chi \rangle} = \frac{|\partial_E S_{n,k}|}{A(S_{n,k})} \frac{A(S_{n,k})}{|S_{n,k}|} \sim \frac{c}{N_n^{1-\gamma} + c} N_n^\gamma (N_n^{1-\gamma} + c) = c N_n^\gamma.$$

The left hand side might be related to $\inf \sigma(\Delta_{B_k})$. Moreover the first factor after the equal sign might be related to α_{B_k} . If this relation holds true we can control the growth and the decrease of these terms by γ . For instance $\inf \sigma(\Delta_{B_k})$ would increase to infinity although α_{B_k} tends to zero for $\gamma < 1$.

We denote for a vertex $v \in B_k$

$$\deg_\pm(v) = |\{w \in S_{k\pm 1} \mid v \sim w\}|,$$

where we set $S_k = B_k \setminus B_{k-1}$ for $k \geq 2$. To prove the theorem we will need the following three Lemmata. In [DKa] the Lemma 1.15 and its subsequent remark gives the following.

Lemma 2. *Let G be a graph. If $(\deg_+(v) - \deg_-(v))/\deg(v) \geq C$ for all $v \in B_n^c$ then $\alpha_{B_n} \geq C$.*

With the help of this Lemma we will prove the statements for α_∞ on the respective graphs.

Lemma 3. *Let $c \geq 1$.*

1. If $\gamma < 1$ then $\alpha_\infty = 0$.
2. If $\gamma = 1$ then $\alpha_\infty = \frac{c}{1+c}$.
3. If $\gamma > 1$ then $\alpha_\infty = 1$.

Proof We get an estimate from above by calculating

$$\alpha_{B_{n-1}} \leq \frac{|\partial_E S_n|}{A(S_n)} = \frac{N_n[cN_n^\gamma] + N_n}{N_n(N_n - 1) + N_n[cN_n^\gamma] + N_n} = \frac{N_n[cN_n^\gamma] + N_n}{N_n^2 + N_n[cN_n^\gamma]}.$$

To obtain a lower bound for $\alpha_{B_{n-1}}$ we use Lemma 2 and calculate

$$\inf_{v \in B_{n-1}^c} \frac{\deg_+(v) - \deg_-(v)}{\deg(v)} = \inf_{k \geq n} \frac{[cN_k^\gamma] - 1}{1 + N_k + [cN_k^\gamma]} = \frac{[cN_n^\gamma] - 1}{1 + N_n + [cN_n^\gamma]}.$$

One gets the desired result by letting n tend to infinity. \square

The next lemma is crucial to show absence of essential spectrum for Δ when $\gamma > 0$.

Lemma 4. *Let $\gamma > 0$ and φ_k functions in $c_c(B_{k-1}^c)$ such that $\|\varphi_k\| \leq 1$ and $\langle \Delta_{B_{k-1}} \varphi_k, \varphi_k \rangle \leq C$ for all $k \in \mathbb{N}$ and some constant $C > 0$. Then*

$$\lim_{k \rightarrow \infty} \|\varphi_k\| = 0.$$

Proof Choose φ_k , $k \in \mathbb{N}$ as assumed. Denote by $\varphi_k^{(i)}$ the restriction of φ_k to $S_i = B_i \setminus B_{i-1}$ for $i \geq k$ and choose $m > k$ such that $\text{supp } \varphi_k \subseteq B_m$. Then an estimate on the form of $\Delta_{B_{k-1}}$ reads

$$\begin{aligned} \langle \Delta_{B_{k-1}} \varphi_k, \varphi_k \rangle &\geq \sum_{i=k}^m \sum_{v \in S_i} \sum_{w \in S_{i+1}, w \sim v} |\varphi_k(v) - \varphi_k(w)|^2 \\ &\geq \sum_{i=k}^m \left(\sum_{v \in S_i} [cN_i^\gamma] \varphi_k^2(v) + \sum_{w \in S_{i+1}} \varphi_k^2(w) \right. \\ &\quad \left. - 2 \sum_{v \in S_i} \sum_{w \in S_{i+1}, w \sim v} \varphi_k(v) \varphi_k(w) \right) \\ &\geq \sum_{i=k}^m \left([cN_i^\gamma] \sum_{v \in S_i} \varphi_k^2(v) + \sum_{w \in S_{i+1}} \varphi_k^2(w) \right. \\ &\quad \left. - 2 \left([cN_i^\gamma] \sum_{v \in S_i} \varphi_k^2(v) \right)^{\frac{1}{2}} \left(\sum_{w \in S_{i+1}} \varphi_k^2(w) \right)^{\frac{1}{2}} \right) \\ &= \sum_{i=k}^m \left([cN_i^\gamma]^{\frac{1}{2}} \|\varphi_k^{(i)}\| - \|\varphi_k^{(i+1)}\| \right)^2 \end{aligned}$$

In the second step we used that each vertex in S_i is uniquely adjacent to $[cN_i^\gamma]$ vertices in S_{i+1} for $k \leq i \leq m$ and in the third step we used the Cauchy-Schwarz inequality. We assumed $\langle \Delta_{B_{k-1}} \varphi_k, \varphi_k \rangle \leq C$ and in particular this is true for every term in sum we estimated above. Moreover $\|\varphi_k^{(i+1)}\| \leq \|\varphi_k\| \leq 1$ for $k \leq i \leq m$ and thus

$$\|\varphi_k^{(i)}\| \leq \frac{\sqrt{C} + \|\varphi_k^{(i+1)}\|}{cN_i^{\frac{\gamma}{2}}} \leq \frac{\sqrt{C} + 1}{cN_i^{\frac{\gamma}{2}}}.$$

Set $C_0 = (\sqrt{C} + 1)/c$. Since the sequence $(N_i^{-\frac{\gamma}{2}})$ is summable we deduce

$$\|\varphi_k\| \leq \sum_{i=k}^m \|\varphi_k^{(i)}\| \leq C_0 \sum_{i=k}^m N_i^{-\frac{\gamma}{2}} \leq C_0 \sum_{i=k}^{\infty} N_i^{-\frac{\gamma}{2}} < \infty.$$

We now let k tend to infinity and conclude $\lim_{k \rightarrow \infty} \|\varphi_k\| = 0$. \square

Proof of Theorem 6. From Proposition 1, 2 and 3 we can deduce

$$1 - \sqrt{1 - \alpha_\infty^2} \leq \inf \sigma_{\text{ess}}(\tilde{\Delta}) \leq \alpha_\infty.$$

Thus by Lemma 3 we get the assertion for α_∞ and $\inf \sigma_{\text{ess}}(\tilde{\Delta})$.

If $\gamma = 0$ we get for the characteristic function $\chi = \chi_{S_{n,k}}$ of $S_{n,k} = B_n \setminus B_k$, $n > k$

$$\frac{\langle \Delta_{B_k} \chi, \chi \rangle}{\langle \chi, \chi \rangle} = \frac{[c](N_k + N_n)}{\sum_{i=k+1}^n N_i} = \frac{[c](\frac{N_k}{N_n} + 1)}{1 + \sum_{i=k+1}^{n-1} \frac{N_i}{N_n}}.$$

Hence by taking the limit over n we have by Proposition 1 that $\inf \sigma_{\text{ess}}(\Delta) \leq [c]$.

Let $\gamma > 0$ and let φ_k be functions in $c_c(B_{k+1}^c)$ such that $\|\varphi_k\| = 1$ and

$$\lim_{k \rightarrow \infty} \langle \Delta_{B_{k+1}} \varphi_k, \varphi_k \rangle = \inf \sigma_{\text{ess}}(\Delta).$$

This is possible by Proposition 1 and a diagonal sequence argument. As $\|\varphi_k\| = 1$ by Lemma 4 the term $\langle \Delta_{B_k} \varphi_k, \varphi_k \rangle$ tends to infinity. Thus the essential spectrum of Δ is empty. \square

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FAKULTÄT FÜR MATHEMATIK, TU CHEMNITZ, D - 09107 CHEMNITZ, GERMANY

CURRENTLY: DEPARTMENT OF MATHEMATICS, FINE HALL PRINCETON UNIVERSITY PRINCETON, NJ 08544, USA

E-mail address: `matthias.keller@mathematik.tu-chemnitz.de`